Tutorial 3 : Selected problem of Assignment 3

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Q1) (HW3 Ex.4) (Reference: Fourier Analysis: an Introduction by Stein-Shakarchi)

Let $f:[-\pi,\pi]\to\mathbb{R}$ be a 2π-periodic integrable function with Fourier coefficients $a_n, b_n \in \mathbb{R}$.

For each $0\le r<1$, define an infinite series of functions by

$$f_r(x) := a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos nx + b_n \sin nx)$$

(a) Show that $f_r(x)$ defines a 2π-periodic continuous function.

(b) Show that $$f_r(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(z)\, f(x-z)\, dz$$

where $$P_r(z) = \frac{1-r^2}{1-2r\cos z + r^2}$$ is called the Poisson kernel.

(c) If in addition f is continuous at x, then

$$\lim_{r\to 1^-} f_r(x) = f(x)$$

Sol: (a) Recall that $r^n(\cos nx + i\sin nx) = r^n e^{inx}$

$$\therefore \quad r^n \cos nx = \frac{r^n e^{inx} + \overline{r^n e^{inx}}}{2} = \frac{r^n e^{inx} + r^n e^{-inx}}{2}$$

Similarly $$r^n \sin nx = \frac{r^n e^{inx} - r^n e^{-inx}}{2i}$$

Formally: $$f_r(x) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos nx + b_n \sin nx)$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \cdot r^n \left(\frac{e^{inx} + e^{-inx}}{2} \right) + b_n r^n \left(\frac{e^{inx} - e^{-inx}}{2i} \right) \right)$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} c_n e^{inx}, \quad \exists\, c_n \in \mathbb{C}$$

$\therefore$ suffice to show $\sum_{n=-\infty}^{\infty} r^{|n|} c_n e^{inx}$ converge uniformly on $\mathbb{R}$:

$$\forall n \in \mathbb{Z}, \quad |c_n| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny}\, dy \right| \leq M, \quad \exists M \in \mathbb{R}$$

$$\therefore \forall x \in \mathbb{R}, \quad \left| r^{|n|} c_n e^{inx} \right| \leq M r^{|n|},$$

and $$\sum_{n=-\infty}^{+\infty} r^{|n|} = 1 + 2 \sum_{m=1}^{\infty} r^m < +\infty \quad \text{as } r < 1$$

$\therefore$ By M-test, $f_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} c_n e^{inx}$ converges uniformly on $\mathbb{R}$

and hence is 2π-periodic continuous as so is true for all

$r^{|n|} c_n e^{inx}$.

(b) $f_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} c_n e^{inx} = \sum_{n=-\infty}^{\infty} r^{|n|} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy \right) e^{inx}$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{-iny} e^{inx} \right) dy$$

(by uniform convergence of $\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} |f(y) r^{|n|} e^{-iny} e^{inx}| \right) dy$)

$$= \frac{1}{2\pi} \int_{x+\pi}^{x-\pi} f(x-z) \left(\sum_{n=-\infty}^{+\infty} r^{|n|} e^{inz} \right) (-dz)$$ (by change of variable $z = x-y$)

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-z) \left(\sum_{n=-\infty}^{+\infty} r^{|n|} e^{inz} \right) dz$$

Therefore, it suffices to estabilish the following identity:

Lemma: $P_r(z) := \dfrac{1-r^2}{1-2r\cos z + r^2} = \sum_{n=-\infty}^{+\infty} r^{|n|} e^{inz}$

Pf of Lemma: Let $\omega := re^{iz}$, then RHS $= \sum_{n=0}^{\infty} \omega^n + \sum_{n=1}^{\infty} \bar{\omega}^n$

$$= \frac{1}{1-\omega} + \bar{\omega} \cdot \frac{1}{1-\bar{\omega}} = \frac{(1-\bar{\omega}) + \bar{\omega}(1-\omega)}{(1-\omega)(1-\bar{\omega})} = \frac{1-|\omega|^2}{(1-\omega)(1-\bar{\omega})}$$

$$= \frac{1-r^2}{(1-re^{iz})(1-re^{-iz})} = \frac{1-r^2}{1-2r\cos z + r^2}$$ —▨

$$\therefore \quad f_r(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(z) f(x-z)\, dz$$

(c) We first prove the following three properties of $\{P_r(z)\}_{r\to 1^-}$:

(P1) $\forall\, 0\le r<1,\ \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(z)\,dz = 1$

(P2) $\exists K\in\mathbb{R}$ s.t. $\forall\, 0\le r<1,\ \frac{1}{2\pi}\int_{-\pi}^{\pi} |P_r(z)|\,dz \le K$

(P3) $\forall\, 0<\delta<\pi,\ \lim_{r\to 1^-}\int_{\delta\le|z|\le\pi} |P_r(z)|\,dz = 0$

Proof of (P1): $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(z)\,dz = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{+\infty} r^{|n|}e^{inz}\right)dz$

$= \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} r^{|n|}\int_{-\pi}^{\pi} e^{inz}\,dz$ (by absolute convergence of $\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{+\infty}|r^{|n|}e^{inz}|\,dz$)

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} dz$ ($\because \forall\, n\ne 0,\ \int_{-\pi}^{\pi} e^{inz}\,dz = 0$) $= 1$ $-\blacksquare$

Proof of (P2) Note that $P_r(z) = \frac{1-r^2}{1-2r\cos z+r^2} > 0$ for all $0\le r<1$ and $-\pi\le z\le\pi$

$\therefore$ Choose $K=1$, then (P1) implies $\frac{1}{2\pi}\int_{-\pi}^{\pi}|P_r(z)|\,dz = 1 = K$ $-\blacksquare$

Proof of (P3) For each $0<\delta<\pi$, let $C_\delta := 1-\cos\delta > 0$, then

$1-2r\cos z+r^2 = (1-r)^2 + 2r(1-\cos z) \ge 0 + 1\cdot(1-\cos\delta) = C_\delta > 0$

$\forall\, \frac{1}{2}\le r<1,\ \delta\le|z|\le\pi$. $\therefore |P_r(z)| \le (1-r^2)\cdot\frac{1}{C_\delta}$

$\therefore \lim_{r\to 1^-}\frac{1}{2\pi}\int_{\delta\le|z|\le\pi}|P_r(z)|\,dz \le \lim_{r\to 1^-}\frac{1}{2\pi}\cdot\left((1-r^2)\cdot\frac{1}{C_\delta}\right)\cdot 2(\pi-\delta) = 0$ $-\blacksquare$

Proof of $\lim_{r\to 1^-} f_r(x) = f(x)$ if f is continuous at x:

Given $\varepsilon>0$, by continuity of f at x, there exists $\delta>0$ such that

$$|f(x-z)-f(x)|<\varepsilon, \quad \forall\ |z|\leq\delta$$

Try to estimate $f_r(x)-f(x)$: $f_r(x)-f(x)$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(z) f(x-z)\,dz - \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(z) f(x)\,dz \quad \text{(by (b) and (P1))}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(z)\big(f(x-z)-f(x)\big)\,dz$$

$$\therefore\ |f_r(x)-f(x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |P_r(z)|\,|f(x-z)-f(x)|\,dz$$

$$= \frac{1}{2\pi}\int_{|z|\leq\delta} |P_r(z)|\,|f(x-z)-f(x)|\,dz + \frac{1}{2\pi}\int_{\delta\leq|z|\leq\pi} |P_r(z)|\,|f(x-z)-f(x)|\,dz$$

$$= \mathrm{I} + \mathrm{II}$$

For I: $\forall r$, $\mathrm{I} \leq \frac{1}{2\pi}\int_{|z|\leq\delta} |P_r(z)|\cdot\varepsilon\,dz \leq \frac{\varepsilon}{2\pi}\cdot\int_{-\pi}^{\pi}|P_r(z)|\,dz \leq \frac{\varepsilon K}{2\pi}$

(by (P2))

For II: $\mathrm{II} \leq \frac{2\|f\|_\infty}{2\pi}\int_{\delta\leq|z|\leq\pi} |P_r(z)|\,dz < \varepsilon$, $\exists\, r_0<1$, $\forall\, r_0\leq r<1$

(by (P3))

$$\therefore\ \forall\, r_0\leq r<1,\ |f_r(x)-f(x)| \leq \mathrm{I}+\mathrm{II} \leq \frac{\varepsilon K}{2\pi}+\varepsilon = \varepsilon\left(\frac{K}{2\pi}+1\right)$$

$$\therefore\ \lim_{r\to 1^-}|f_r(x)-f(x)| = 0, \quad \text{i.e. } \lim_{r\to 1^-} f_r(x) = f(x) \qquad \text{—}\ \square$$

Rmk: (1) (P1)-(P3) says that $\{P_r(z)\}_{r\to 1^-}$ is a "good kernel"

which in particular ensures that (c) is true.

(2) Dirichlet kernel $\{D_N(z)\}_{N\to\infty}$ satisfies (P1), (P3) but NOT (P2):

In fact, Property IV in Lecture Note Ch. I says the contrary:

$$\forall \delta>0, \int_{|z|\le\delta} |D_n(z)|\,dz \to \infty \quad \text{as} \quad n\to\infty$$

Therefore, estimate in I fails, unless f has some decay condition at x

e.g. f is Lipschitz continuous at x.